

Triangles

- **Similar and Congruent Figures**

- Two geometric figures having the same shape and size are said to be congruent figures.
- Two geometric figures having the same shape, but not necessarily the same size, are called similar figures.

Example:

- (1) All circles are similar.
- (2) All equilateral triangles are similar.
- (3) All congruent figures are similar. However, the converse is not true.

- **Similarity of Polygons**

Two polygons with the same number of sides are similar, if

- their corresponding angles are equal
- their corresponding sides are in the same ratio (or proportion)

- Two line segments are congruent, if they are equal in length.
- Two angles are congruent, if they have the same measure.

- **CPCT:**

CPCT stands for Corresponding Parts of Congruent Triangles.

If $\triangle ABC \cong \triangle PQR$, then corresponding sides are equal i.e., $AB = PQ$, $BC = QR$, and $CA = RP$ and corresponding angles are equal i.e., $\angle A = \angle P$, $\angle B = \angle Q$, and $\angle C = \angle R$.

- **Basic proportionality theorem:**

If a line is drawn parallel to one side of a triangle to intersect the other two sides in distinct points, then the other two sides are divided in the same ratio.

Corollary: If D and E are points on the sides, AB and AC, respectively of $\triangle ABC$ such that $DE \parallel BC$, then

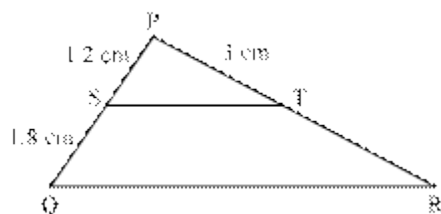
$$\frac{AB}{AD} = \frac{AC}{AE}$$

$$\frac{AB}{DB} = \frac{AC}{EC}$$

Example:

In the given figure, S and T are points on PQ and PR respectively of $\triangle PQR$ such that $ST \parallel QR$.





Determine the length of PR.

Solution:

Since $ST \parallel QR$, by basic proportionality theorem, we have

$$\frac{PS}{SQ} = \frac{PT}{TR}$$

$$\Rightarrow \frac{1.2}{1.8} = \frac{3}{TR}$$

$$\Rightarrow TR = \frac{3 \times 1.8}{1.2} = 4.5 \text{ cm}$$

$$\therefore PR = PT + TR = (3 + 4.5) \text{ cm} = 7.5 \text{ cm}$$

- **Converse of basic proportionality theorem:**

If a line divides any two sides of a triangle in the same ratio, then the line is parallel to the third side.

- **Theorem: (AAA similarity criterion)**

If in two triangles, corresponding angles are equal, then their corresponding sides are in the same ratio (or proportion) and hence, the two triangles are similar.

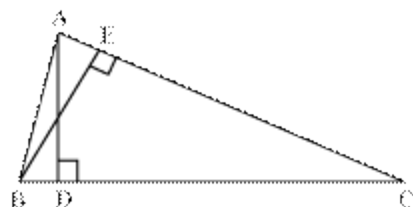
- **Theorem: (AA similarity criterion)**

If in two triangles, two angles of one triangle are respectively equal to the two angles of the other triangle, then the two triangles are similar.

Example:

In $\triangle ABC$, $\angle C$ is acute, D and E are points on sides BC and AC respectively, such that $AD \perp BC$ and $BE \perp AC$. Show that $BC \times CD = AC \times CE$.

Solution:



In $\triangle ADC$ and $\triangle BEC$,
 $\angle ADC = \angle BEC = 90^\circ$

$\angle DCA = \angle ECB$ [Common]
 \therefore By AA similarity criterion, $\triangle ADC \sim \triangle BEC$

$$\therefore \frac{CD}{CE} = \frac{AC}{BC}$$

$$\Rightarrow BC \times CD = AC \times CE$$

Hence, the result is proved.

- Theorem: (SSS similarity criterion)**

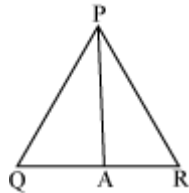
If in two triangles, sides of one triangle are proportional to the sides of the other triangle then the two triangles are similar by SSS similarity criterion.

Example:

If PQR is an isosceles triangle with $PQ = PR$ and A is the mid-point of side QR then prove that $\triangle PAQ$ is similar to $\triangle PAR$.

Solution:

It is given that $\triangle PQR$ is an isosceles triangle and $PQ = PR$.



In triangles PAQ and PAR,

$$PQ = PR$$

Also, A is the mid-point of QR, therefore $QA = AR$

And, $PA = PA$ (Common to both triangles)

Therefore, we can say that

$$\frac{PQ}{PR} = \frac{QA}{AR} = \frac{PA}{PA}$$

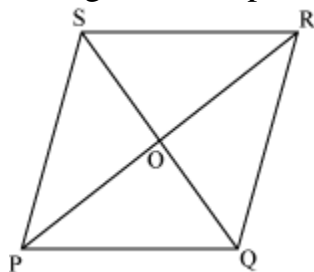
\therefore Using SSS similarity criterion, we obtain $\triangle PAQ \sim \triangle PAR$

- Theorem: (SAS similarity criterion)**

If one angle of a triangle is equal to one angle of the other triangle and the sides including these angles are proportional, then the two triangles are similar by SAS similarity criterion.

Example:

If PQRS is a parallelogram, then prove that $\triangle SOR$ is similar to $\triangle POQ$.



Solution:

Consider $\triangle SOR$ and $\triangle POQ$.

Since PQRS is a parallelogram, the diagonals bisect each other.

$\therefore SO = OQ$ and $PO = OR$

and $\angle POQ = \angle SOR$ (Vertically opposite angles)

By SAS similarity criterion, we obtain

$\triangle SOR \sim \triangle QOP$

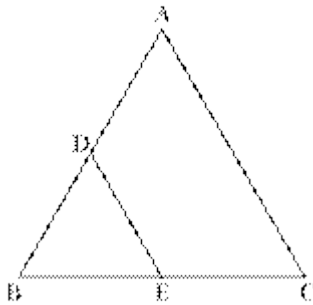
- Areas of similar triangles**

Theorem: The ratio of the areas of two similar triangles is equal to the square of the ratio of their corresponding sides.

Example:

In $\triangle ABC$, D and E are the respective mid-points of sides AB and BC. Find the ratio of the areas of $\triangle DBE$ and $\triangle ABC$.

Solution:



In $\triangle ABC$, D and E are the respective mid-points of the sides, AB and BC.

By the converse of BPT, $DE \parallel AC$

In $\triangle DBE$ and $\triangle ABC$,

$\angle DBE = \angle ABC$ [Common]

$\angle BED = \angle BCA$ [Corresponding angles]

$\angle BDE = \angle BAC$ [Corresponding angles]

\therefore By AAA similarity criterion, $\triangle DBE \sim \triangle ABC$

$$\begin{aligned} \Rightarrow \frac{\text{Area}(\triangle DBE)}{\text{Area}(\triangle ABC)} &= \left(\frac{BE}{BC}\right)^2 \\ &= \left(\frac{BE}{2BE}\right)^2 \quad [\text{E is mid-point of BC}] \\ &= \frac{1}{4} \\ &= 1:4 \end{aligned}$$

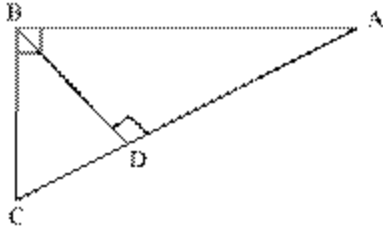
Result: Using the above theorem, the result “ the ratio of the areas of two similar triangles is equal to the square of the ratio of their corresponding medians or altitudes or angle bisector” can be proved.

- Pythagoras theorem:**

In a right triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.

Example:

$\triangle ABC$ is right-angled at B and $BD \perp CA$.



Prove that $BD^2 = CD \times DA$.

Solution:

By applying Pythagoras theorem in $\triangle BDC$, $\triangle BDA$, and $\triangle ABC$, we obtain

$$BC^2 = CD^2 + BD^2 \quad \dots (1)$$

$$BA^2 = BD^2 + DA^2 \quad \dots (2)$$

$$CA^2 = BC^2 + BA^2 \quad \dots (3)$$

Adding equations (1) and (2), we obtain

$$BC^2 + BA^2 = 2BD^2 + CD^2 + DA^2$$

$$\Rightarrow CA^2 = 2BD^2 + CD^2 + DA^2 \quad \dots \text{[Using (3)]}$$

$$\Rightarrow (CD + DA)^2 = 2BD^2 + CD^2 + DA^2$$

$$\Rightarrow CD^2 + DA^2 + 2 \times CD \times DA = 2BD^2 + CD^2 + DA^2$$

$$\Rightarrow CD \times DA = BD^2$$

- **Converse of Pythagoras theorem:**

In a triangle, if the square of one side is equal to the sum of the squares of other two sides, then the angle opposite to the first side is a right angle.